

Dual Instantons

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Abstract

We show how to map the Belavin-Polyakov instantons of the $O(3)$ -nonlinear σ -model to a dual theory where they then appear as nontopological solitons. They are stationary points of the Euclidean action in the dual theory, and moreover, the dual action and the $O(3)$ -nonlinear σ -model action agree on shell.

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Although many techniques have now been devised for finding dual descriptions of field theories, important questions and limitations remain. (For reviews see[1].) One limitation is that most of the techniques are applicable only in two space-time dimensions. Within the realm of nonabelian T-duality, there are issues concerning the global aspects of the theory. T-dual theories are equivalent at the level of the classical dynamics, and also to several orders in perturbation theory. Moreover, from the current algebras, the dual descriptions are known to be canonically equivalent. But canonical equivalence is insufficient for proving the equivalence of Feynman path integrals.[2] More troubling is the result that the canonical equivalence between dual theories is, in general, only valid locally, as the configuration spaces of the theories can have different topological properties. We can have that certain solutions are ‘topological’ in one theory, but not in its dual, as we shall demonstrate here. Could this lead to a breakdown in the quantum equivalence of the two theories?

The example we shall look at is that of the $O(3)$ nonlinear σ -model, having a target space of S^2 . From the condition of finite action in two-dimensional Euclidean space-time, one gets that the configuration space is a union of disjoint pieces, and well known topological solitons appear, namely the Belavin-Polyakov instantons.[3] The Bogomol’nyi bound insures that these solutions are the minima in every topological sector of the theory.[4] Recently, a Lagrangian and Hamiltonian field theory was found which is *locally* canonically equivalent to the $O(3)$ nonlinear σ -model, and is generalizable to nonlinear G/H models for any Lie groups G and $H \subset G$. [5] * However, the target space for this dual theory is topologically trivial, and finite action restrictions do not lead to any disconnected regions of the configuration space. (Of course, at the classical level the dual action can only be determined up to divergence terms since one only demands equivalence of equations of motion. But there are no divergence terms that can be added to the dual Euclidean action for the purpose of obtaining a nontrivial topology, and moreover the dual Euclidean action cannot even be made to be bounded from below.) Thus the Belavin-Polyakov instantons must appear as non-topological classical solutions in the dual theory, where their stability is not automatically assured. On the other hand, here we show that our dual action agrees on shell with the action of the $O(3)$ nonlinear σ -model. (In the appendix we generalize this result to G/H -models.) It then follows that, on shell, the dual Euclidean action is bounded from below, and that classical solutions of the dual theory satisfy the Bogomol’nyi bound. It also leads to a *dynamical* flux quantization condition. For answering the question in the first paragraph, a semiclassical path integral can be computed in the dual theory and compared with analogous calculations for the $O(3)$ nonlinear σ -model. (For example, one can try to

*It is also generalizable to dynamics consistent with Poisson-Lie T-duality[6], [7],[5].

reproduce the results of Fateev, Frolov and Schwarz[8] in the dual theory.) We plan to report on these calculations in a forthcoming article.

In this letter, we give an explicit construction of the instantons of the dual theory.[†] The construction involves gluing Belavin-Polyakov instantons together with corresponding anti-instantons of opposite winding number. The dual instantons are seen to have one zero mode which is not present for Belavin-Polyakov instantons.

We first review the $O(3)$ nonlinear σ -model, which we shall refer to as the primary theory, and its dual formulation[5]. The target space for the $O(3)$ nonlinear σ -model is S^2 , which is span by the fields $\psi^i(x)$, $i = 1, 2, 3$, $\psi^i(x)\psi^i(x) = 1$. We shall specialize to two-dimensional Euclidean space-time. The standard Lagrangian density L is

$$L = \frac{\kappa}{2} \partial_\mu \psi^i \partial_\mu \psi^i , \quad (1)$$

where κ is the coupling constant. This system can also be re-expressed in terms of $SU(2)$ -valued fields $g(x)$. [10],[11] We take g to be in the defining representation and write

$$\psi^i(x) \sigma^i = g(x) \sigma^3 g(x)^{-1} , \quad (2)$$

σ_i being the Pauli matrices. This introduces an additional $U(1)$ gauge degree of freedom, associated with $g(x) \rightarrow g(x) \exp -\frac{i}{2} \lambda(x) \sigma^3$. The corresponding $U(1)$ connection is

$$A_\mu = i \operatorname{Tr} \sigma_3 (g^{-1} \partial_\mu g) . \quad (3)$$

In addition, one can introduce a complex current

$$\Pi_\mu = i \epsilon_{\mu\nu} \operatorname{Tr} \sigma^+ (g^{-1} \partial_\nu g) , \quad \sigma^+ = \sigma^1 + i \sigma^2 , \quad (4)$$

which rotates in the complex plane under a gauge transformation. The gauge invariant Lagrangian may be re-expressed in terms of these currents as

$$L = \frac{\kappa}{2} |\Pi_\mu|^2 . \quad (5)$$

The equations of motion resulting from variations of g , $\delta g = -\frac{i}{2} g \sigma^i \epsilon^i$, ϵ^i being infinitesimal, state that the covariant curl of Π is zero:

$$\epsilon_{\mu\nu} D_\mu \Pi_\nu = 0 . \quad (6)$$

where the covariant derivative is defined by $D_\mu \Pi_\nu = \partial_\mu \Pi_\nu + i A_\mu \Pi_\nu$. Along with the equations of motion (6), we have three identities. These are just the Maurer-Cartan equations, which in terms of A_μ and Π_μ , are

$$D_\mu \Pi_\mu = 0 , \quad (7)$$

$$F = -\frac{i}{2} \epsilon_{\mu\nu} \Pi_\mu \Pi_\nu^* , \quad (8)$$

[†]For dual instantons in gravity see [9].

F being the $U(1)$ curvature, $F = \epsilon_{\mu\nu} \partial_\mu A_\nu$. Finite action requires that we identify the points at infinity and compactify the Euclidean space-time to S^2 . The configuration space is then a union of disconnected sectors associated with $\Pi_2(S^2)$. F is proportional to the winding number density

$$\rho = \frac{1}{8\pi} \epsilon^{ijk} \epsilon_{\mu\nu} \psi^i \partial_\mu \psi^j \partial_\nu \psi^k, \quad (9)$$

and the total flux is

$$\int_{S^2} d^2x F = 4\pi n, \quad (10)$$

n being the winding number.

The dual Lagrangian \tilde{L} in Minkowski space was specified in [5]. It was given in terms of a complex scalar field χ and the $U(1)$ connection A_μ (now regarded as independent field variables). It was useful to also introduce an auxiliary scalar θ . One can perform a Wick rotation to obtain the corresponding Euclidean action. We specify the Wick rotation later, and for now just assume the Euclidean Lagrangian \tilde{L} to have the general form

$$\begin{aligned} \tilde{L} &= \tilde{L}_0 + L_{BF}, \\ \tilde{L}_0 &= \frac{\alpha}{2} |D_\mu \chi|^2 + \frac{i\beta}{2} \epsilon_{\mu\nu} (D_\mu \chi) (D_\nu \chi)^*, \quad L_{BF} = \theta F. \end{aligned} \quad (11)$$

$\alpha \delta^{ab}$ and $\beta \epsilon^{ab}$ represent the dual metric and antisymmetric tensor, respectively. The covariant derivative $D_\mu \chi$ is defined by $D_\mu \chi = \partial_\mu \chi + i A_\mu \chi$. Under gauge transformations,

$$\chi \rightarrow e^{-i\lambda(x)} \chi, \quad A_\mu \rightarrow A_\mu + \partial_\mu \lambda, \quad (12)$$

while θ is assumed to be gauge invariant, and thus so is L_{BF} . Like in [5], we will assume that α and β are independent of χ and A , and hence \tilde{L}_0 is gauge invariant. On the other hand, we allow for a nontrivial dependence on θ . The expression for these functions is given below. For the dual theory to correspond to the primary theory, we should compactify the Euclidean space-time manifold to S^2 . Then, in general, A_μ is not globally defined, i.e. the curvature two-form is closed but not exact.

Following [5], it is easy to show that we recover the equations of the primary theory, i.e., (6), (7) and (8), starting from the dual Lagrangian (11), for a certain α and β . Furthermore, although \tilde{L} is not positive definite, we show that $\tilde{S} = \int_{S^2} d^2x \tilde{L} \geq 0$ on shell, and moreover that its numerical value is identical to that of the primary Euclidean action $S = \int_{S^2} d^2x L$.

We first reproduce the equations (6), (7) and (8). For the equations of motion resulting from variations of χ , we can ignore the BF -term. From the assumption that α and β are independent of χ , we easily recover (7), once we define Π_μ according to

$$\Pi_\mu = -\alpha D_\mu \chi + i\beta \epsilon_{\mu\nu} D_\nu \chi. \quad (13)$$

This definition leads to the identity

$$\text{Im } D_\mu \chi \Pi_\mu^* = 0 , \quad (14)$$

and upon using the equations of motion (7), it follows that $\epsilon_{\mu\nu} \text{Im } (\chi \Pi_\nu^*) dx^\mu$ is a closed one-form. From variations of A in \tilde{L} , it is also exact:

$$\epsilon_{\mu\nu} \partial_\nu \theta = -\text{Im } \chi \Pi_\mu^* . \quad (15)$$

Variations of θ in \tilde{L} lead to

$$F = -\frac{\alpha'}{2} |D_\mu \chi|^2 - \frac{i\beta'}{2} \epsilon_{\mu\nu} (D_\mu \chi) (D_\nu \chi)^* , \quad (16)$$

the prime indicating a derivative with respect to θ . This agrees with (8) provided that

$$\alpha' = 2\alpha\beta , \quad \beta' = (\beta^2 + \alpha^2) . \quad (17)$$

These equations are solved by

$$\alpha = -\frac{\kappa}{\kappa^2 - \theta^2} , \quad \beta = \frac{\theta}{\kappa^2 - \theta^2} , \quad (18)$$

up to a constant translation in θ . Eq. (16) is then a fourth order equation for θ which, in principle, can be used to eliminate the auxiliary scalar field. κ denotes the coupling constant of the dual theory. From the Hamiltonian analysis of the Minkowski formulation of this system, it is identical to the coupling constant κ of the primary theory. It remains to obtain (6). For this we need another identity, which is obtained by inverting (13), using (18), to solve for $D_\mu \chi$:

$$D_\mu \chi = \kappa \Pi_\mu - i\theta \epsilon_{\mu\nu} \Pi_\nu . \quad (19)$$

Now take the covariant curl to get $-i\kappa \epsilon_{\mu\nu} D_\mu \Pi_\nu = F \chi - \partial_\mu \theta \Pi_\mu - \theta D_\mu \Pi_\mu$. The right hand side vanishes upon imposing the equations of motion (7), (8) and (15), and hence we recover the equation of motion of the primary formulation (6).

By comparing (11) with the Lagrangian in [5] [where we assume the metric tensor $\text{diag}(1, -1)$], we see that the Wick rotation from Minkowski to Euclidean space-time affects scalar as well as vector fields:

$$\begin{aligned} \partial_0 &\rightarrow i\partial_0 , & A_0 &\rightarrow iA_0 , \\ \theta &\rightarrow i\theta , & \chi &\rightarrow i\chi , & \chi^* &\rightarrow i\chi^* . \end{aligned} \quad (20)$$

(We also added a total divergence to the Lagrangian in [5].)

The dual Lagrangian (11) can be re-expressed in several curious ways. One way is to substitute the definition of Π_μ in (13) back into \tilde{L}_0 and integrate by parts to get

$$\tilde{L}_0 = \chi^* D_\mu \Pi_\mu - \partial_\mu (\chi^* \Pi_\mu) . \quad (21)$$

Reality follows from (14). The second term gives no contribution to the action for the domain S^2 . (For this note that $\chi^* \Pi_\mu$ is globally defined.) Moreover, the first term, and hence the action $\tilde{S}_0 = \int_{S^2} d^2x \tilde{L}_0$, vanishes when evaluated on the space of classical solutions, which we denote by $\tilde{S}_0|_{cl} = 0$. Alternatively, we can write \tilde{L}_0 quadratically in terms of the currents if we substitute (19) back into (11),

$$\tilde{L}_0 = -\frac{\kappa}{2} |\Pi_\mu|^2 - \frac{i\theta}{2} \epsilon^{\mu\nu} \Pi_\mu \Pi_\nu^* . \quad (22)$$

The first term is minus the primary Lagrangian (5) upon identifying the coupling constants of the theory. Furthermore, the second term is equivalent to the BF -term after using (8). Other possible forms for \tilde{L}_0 are obtained by taking linear combinations of (21) and (22). Taking twice (21) minus (22) gives

$$\tilde{L} = \frac{\kappa}{2} |\Pi_\mu|^2 + \theta(F + \frac{i}{2} \epsilon^{\mu\nu} \Pi_\mu \Pi_\nu^*) + 2\chi^* D_\mu \Pi_\mu - 2\partial_\mu (\chi^* \Pi_\mu) , \quad (23)$$

where we added the BF -term. This implies that the primary and dual actions coincide on shell,

$$\tilde{S}|_{cl} = \int_{S^2} d^2x \tilde{L} \Big|_{cl} = \frac{\kappa}{2} \int_{S^2} d^2x |\Pi_\mu|^2 \Big|_{cl} = S|_{cl} , \quad (24)$$

and thus the dual action evaluated on the space of classical solutions is positive definite (with the vacuum solution corresponding to vanishing currents Π_μ .) The result that a dual action can be found which agrees on shell with the primary action can be generalized to G/H models for any Lie groups G and $H \subset G$ (see Appendix). The dual action is (in Minkowski space-time) is given in [5].

Although the space of field configurations in the dual version of the $O(3)$ nonlinear σ -model is topologically trivial, (24) implies that the subspace of all classical solutions with finite Euclidean action is a union disconnected regions. The latter are classified by the total flux, which we know from the primary theory is quantized according to (10). We can therefore say that the quantization condition is dynamically generated. It does not appear to result from any kinematic considerations of the dual theory, as, classically, all values of the flux are allowed.[‡] On the other hand, a semiclassical argument based on Wilson loops $W(C) = \exp i \int_C A$ gives flux quantization, but it differs from (10). Demanding that the expectation value of $W(C)$ is independent of the coordinate

[‡]In this regard, note that if the value of α at spatial infinity is restricted to being finite, a bounded Euclidean action does not necessarily imply that A must go to a pure gauge at spatial infinity.

patch chosen on S^2 for any closed path C gives $\int_{S^2} d^2x F = 2\pi \times \text{integer}$. With this quantization condition, which is identical to the Dirac quantization of magnetic charge, we can allow for, say, merons.[12] However, such solutions are known to have infinite Euclidean action.

The instantons and anti-instantons of Belavin and Polyakov[3] are self-dual and anti-self-dual solutions, respectively, and they correspond to the minima of the Euclidean action of the primary theory in every topological sector. They are therefore ‘topologically’ stable. For this one can write L in (5) according to

$$L = \frac{\kappa}{4} |\Pi_\mu \pm i\epsilon_{\mu\nu} \Pi_\nu|^2 \pm \frac{i\kappa}{2} \epsilon_{\mu\nu} \Pi_\mu \Pi_\nu^* \quad (25)$$

The Bogomol’nyi bound[4] for the Euclidean action of the primary theory then follows from (8)

$$S = \int_{S^2} d^2x L \geq 4\pi\kappa|n| ,$$

with the lower bound saturated by self-dual (instanton) configurations, i.e. $\Pi_\mu - i\epsilon_{\mu\nu} \Pi_\nu = 0$ when $n > 0$, and anti-self-dual (anti-instanton) configurations, i.e. $\Pi_\mu + i\epsilon_{\mu\nu} \Pi_\nu = 0$ when $n < 0$. Of course, the instantons (and anti-instantons) are also solutions of the dual theory, and from (24) they have the same value for the action as in the primary theory, i.e. $\tilde{S}|_d = 4\pi\kappa|n|$. However, n cannot represent a topological index in the dual theory, as the target space topology is trivial, and now stability cannot be assured from topology.

Below we construct the general form of the instanton solutions in the dual theory.

We first review the construction of the most general instanton solutions in the primary theory.[3] For this it was found convenient to perform a stereographic projection, and write the scalar fields ψ^i in terms of a complex function $W(x)$,

$$\psi^1 + i\psi^2 = \frac{2W}{1 + |W|^2} , \quad \psi^3 = \frac{|W|^2 - 1}{|W|^2 + 1} . \quad (26)$$

In terms of this function the Lagrangian (1) and instanton number density (9) become

$$L = \frac{4\kappa}{(1 + |W|^2)^2} (|\partial_z W|^2 + |\partial_z W^*|^2) \quad (27)$$

$$\rho = \frac{1}{\pi(1 + |W|^2)^2} (|\partial_z W|^2 - |\partial_z W^*|^2) \quad (28)$$

where we use the complex coordinate $z = x_0 + ix_1$. From (25) instantons require that $L = 4\pi\kappa\rho$. This is only possible for $\partial_{z^*} W = 0$, and therefore W is an analytic function of z . Alternatively, anti-instantons require that $L = -4\pi\kappa\rho$, leading to W being an

analytic function of z^* . For the choice of boundary conditions $W \rightarrow 1$, as $|x| \rightarrow \infty$, the general instanton solution with winding number n has the form[3]

$$W(z) = \frac{\prod_{i=1}^n (z - a_i)}{\prod_{j=1}^n (z - b_j)} , \quad (29)$$

where a_i and b_i are complex constants.

To write down the currents Π_μ and connection one form A associated with the instanton, we must fix a gauge $g(W)$ for the $SU(2)$ -valued field g . In general, this can only be done locally. A gauge choice which is everywhere valid away from the poles of $W(z)$ is

$$g_S(W) = \frac{1}{\sqrt{1 + |W|^2}} \begin{pmatrix} W^* & -1 \\ 1 & W \end{pmatrix} . \quad (30)$$

Alternatively, one that is every valid away from the zeros of $W(z)$ is

$$g_N(W) = \frac{|W|}{\sqrt{1 + |W|^2}} \begin{pmatrix} 1 & -W^{-1} \\ W^{*-1} & 1 \end{pmatrix} . \quad (31)$$

Since the general solution (29) contains poles as well as zeros, we will need to cover S^2 with at least two open regions, R_S^2 containing the zeros and R_N^2 containing the poles. We can then make the gauge choice (30) for R_S^2 , and (31) for R_N^2 .[§] In R_S^2 , we have the left invariant one form

$$g_S^{-1} dg_S = \frac{1}{1 + |W|^2} \begin{pmatrix} \frac{1}{2}(W dW^* - W^* dW) & dW \\ -dW^* & -\frac{1}{2}(W dW^* - W^* dW) \end{pmatrix} , \quad (32)$$

while in R_N^2 the left invariant one form $g_N^{-1} dg_N$ is obtained by simply replacing W by $-W^{-1}$ everywhere in (32). The z components of the currents and $U(1)$ connection are then

$$\begin{aligned} \Pi_z^{(S)} &= \frac{1}{2}(\Pi_0 - i\Pi_1) = 0 \\ \tilde{\Pi}_z^{(S)} &= \frac{1}{2}(\Pi_0^* - i\Pi_1^*) = \frac{-2 \partial_z W}{1 + |W|^2} \\ A_z^{(S)} &= \frac{1}{2}(A_0 - iA_1) = -i \partial_z \ln(1 + |W|^2) , \end{aligned} \quad (33)$$

in R_S^2 , and

$$\begin{aligned} \Pi_z^{(N)} &= \frac{1}{2}(\Pi_0 - i\Pi_1) = 0 \\ \tilde{\Pi}_z^{(N)} &= \frac{1}{2}(\Pi_0^* - i\Pi_1^*) = \frac{2 \partial_z W^{-1}}{1 + |W|^{-2}} \end{aligned}$$

[§]On the other hand, a global gauge exists for solutions containing only zeros, or only poles. This will require that W have boundary value ∞ or 0 , respectively, as $|x| \rightarrow \infty$.

$$A_z^{(N)} = \frac{1}{2}(A_0 - iA_1) = -i \partial_z \ln(1 + |W|^{-2}) \quad (34)$$

in R_N^2 . In both (33) and (34) we used $\partial_{z^*}W = 0$. As stated earlier this is consistent with the condition of self-duality, i.e., $\Pi_z = 0$. It is easy to write down the transition function $\lambda^{(NS)}$ between the two gauges in the overlapping region $R_S^2 \cap R_N^2$:

$$\lambda^{(NS)} = i \ln \frac{W(z)}{W(z)^*} \quad (35)$$

which transforms A_z and $\tilde{\Pi}_z$ according to $A_z^{(N)} = A_z^{(S)} + \partial_z \lambda^{(NS)}$ and $\tilde{\Pi}_z^{(N)} = e^{i\lambda^{(NS)}} \tilde{\Pi}_z^{(S)}$ (The above analysis can easily be repeated for anti-instantons, corresponding to $\tilde{\Pi}_z = 0$.)

Before writing down the instanton solution in the dual theory, we first look at the implications of self-duality and anti-self-duality. Eqs. (15) and (19) for the scalar fields, can be expressed as

$$\begin{aligned} \partial_z \theta &= \frac{1}{2}(\chi \tilde{\Pi}_z - \chi^* \Pi_z) \\ D_z \chi &= \partial_z \chi + iA_z \chi = (\kappa + \theta) \Pi_z \\ D_z \chi^* &= \partial_z \chi^* - iA_z \chi^* = (\kappa - \theta) \tilde{\Pi}_z \end{aligned} \quad (36)$$

Instantons, i.e. $\Pi_z = 0$, imply $D_z \chi = 0$, while anti-instantons, i.e. $\tilde{\Pi}_z = 0$, imply $D_z \chi^* = 0$. In either case, we can then write the connection in terms of scalar fields. Furthermore, from (36), it follows that $(\theta - \kappa)^2 + |\chi|^2$ is a constant for $\Pi_z = 0$, while $(\theta + \kappa)^2 + |\chi|^2$ is a constant for $\tilde{\Pi}_z = 0$. Therefore, when the currents are restricted to being self-dual or anti-self-dual, the scalar fields of the dual theory define a two-sphere, in analogy with the scalar fields of the primary theory. [One major difference with the primary theory, though, is that while ψ^i are gauge invariant, χ is not. χ contains only one gauge invariant degree of freedom, and hence the gauge invariant degrees of freedom in the self-dual or anti-self-dual fields of the dual theory, in fact, span S^1 . As $\Pi_2(S^1) = 0$, the topology of this space is trivial.] We can parametrize the scalar fields in terms of a complex function, say $V(x)$, via a stereographic projection, as was done for the primary theory. In the case of instantons, i.e., $\Pi_z = 0$, we write

$$\begin{aligned} \chi &= \frac{2RV^*}{1 + |V|^2}, \\ \theta &= R \frac{1 - |V|^2}{1 + |V|^2} + \kappa, \end{aligned} \quad (37)$$

where R is the radius of the two-sphere. This expression is valid in any open subset of S^2 . By comparing $D_z \chi = 0$ with the equations of motion $D_z(\tilde{\Pi}_z)^* = 0$, we get that $(\tilde{\Pi}_z)^* = G(z)^* \chi$. G is an analytic function of z , and from the last equation in (36), it is equal to $-\frac{1}{R} \partial_z \ln |V|^2$. Then up to a phase (which can be gauged away) V

is either analytic or anti-analytic in z . In general, both cases are needed for the global description of solutions with non-zero total flux. The global description is obtained by matching solutions in the overlapping regions of different open subsets of S^2 . An easy way to proceed is to use our previous result that finite action solutions of the dual theory correspond to finite action solutions of the primary theory. By identifying the currents and connections of the primary theory (33) and (34) with those derived from (37), we get the gauge choice $V(z) = W(z)$ in R_S^2 , and $V(z^*) = -1/W(z)^*$ in R_N^2 . The transition function is once again given by (35). The integration constant R drops out of the expression for the currents and connections, and hence represents a degeneracy in the space of solutions in the dual theory. This implies that the dual instantons have a zero mode which is not present for the instantons in the primary theory. In matching the solutions for χ and θ in R_S^2 and R_N^2 , we note that θ is gauge invariant. Then if we set $V(z) = W(z)$ in R_S^2 , and $V(z^*) = -1/W(z)^*$ in R_N^2 in (37) we must switch the sign of R in the two regions. The dual instanton solution is thus

$$\begin{aligned}\chi(z) &= -\frac{2RW(z)^*}{1+|W(z)|^2} \quad \text{in } R_S^2, \\ &= -\frac{2RW(z)}{1+|W(z)|^2} \quad \text{in } R_N^2,\end{aligned}\tag{38}$$

$$\theta(z) = R \frac{|W(z)|^2 - 1}{|W(z)|^2 + 1} + \kappa.\tag{39}$$

If $(\psi_{(n)}^1, \psi_{(n)}^2, \psi_{(n)}^3)$ corresponds to the n -instanton solution of the primary theory expressed in terms of the fields ψ^i , then the dual n -instanton solution can be written

$$\left(\frac{\chi^1}{R}, \frac{\chi^2}{R}, \frac{\theta - \kappa}{R}\right) = \begin{cases} (-\psi_{(n)}^1, \psi_{(n)}^2, \psi_{(n)}^3) & \text{in } R_S^2 \\ (-\psi_{(n)}^1, -\psi_{(n)}^2, \psi_{(n)}^3) & \text{in } R_N^2 \end{cases},\tag{40}$$

where $\chi = \chi^1 + i\chi^2$. Thus instantons in the dual theory are obtained by gluing instantons of the primary theory together with anti-instantons of opposite winding number, the latter being obtained by switching the orientation of one of the components.

An analogous result can be found for the anti-instantons of the dual theory. In that case, the right hand side of (40) gets replaced by $(\frac{\chi^1}{R}, \frac{\chi^2}{R}, \frac{\theta + \kappa}{R})$.

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Appendix

Here we show that for any nonlinear G/H model, the dual action [5] agrees on shell with the primary action, up to boundary terms.

Say G and $H \subset G$ are N and $N - M$ dimensional groups, respectively, with the former generated by e_i , $i = 1, 2, \dots, N$, and having commutation relations: $[e_i, e_j] = c_{ij}^k e_k$. We can split the generators into e_a , $a = 1, 2, \dots, M$ and $\hat{e}_\alpha = e_{M+\alpha}$, $\alpha = 1, 2, \dots, N - M$, the latter generating H , $[\hat{e}_\alpha, \hat{e}_\beta] = \hat{c}_{\alpha\beta}^\gamma \hat{e}_\gamma$, $\hat{c}_{\alpha\beta}^\gamma = c_{M+\alpha, M+\beta}^{M+\gamma}$. We will assume that the metric \mathbf{g}_{ij} on G is nondegenerate and block diagonal, i.e. $\mathbf{g}_{a, M+\alpha} = 0$. The structure constants satisfy $c_{M+\alpha, M+\beta}^c = 0$, $c_{M+\alpha, b}^{M+\gamma} = 0$. The second relation follows from the first, using the invariance property $c_{jk}^i \mathbf{g}_{i\ell} = \mathbf{g}_{ji} c_{k\ell}^i$.

In the primary theory, the fundamental fields $g(x)$ have values in G . Utilizing the group metric \mathbf{g}_{ij} projected onto G/H , the primary Lagrangian can be expressed as

$$L = -\frac{\kappa}{2} \mathbf{g}_{ab} (g^{-1} \partial_\mu g)^a (g^{-1} \partial^\mu g)^b, \quad (41)$$

where $a, b = 1, 2, \dots, M$ and κ is the coupling constant. $(g^{-1} dg)^a$ denotes the e_a component of the one-form $g^{-1} dg$. L is gauge invariant under $g(x) \rightarrow g(x)h(x)$, $h(x) \in H$, and consequently defines a theory on G/H . L is also invariant under global transformations $g \rightarrow g_0 g$, $g_0 \in G$. There are now M equations of motion resulting from variations of g , and they can be written as

$$\epsilon^{\mu\nu} (D_\mu \pi_\nu)^a = 0, \quad (42)$$

where

$$\pi_\mu^a = \epsilon_{\mu\nu} (g^{-1} \partial^\nu g)^a, \quad A_\mu^\alpha = (g^{-1} \partial_\mu g)^{M+\alpha}. \quad (43)$$

and the covariant derivative is now defined by $(D_\mu \pi_\nu)^a = \partial_\mu \pi_\nu^a + c_{M+\beta}^a A_\mu^\beta \pi_\nu^c$. A_μ^α transforms as components of an H connection one-form. In addition to the equations of motion (42), we have N Maurer-Cartan equations:

$$(D^\mu \pi_\mu)^a = \frac{1}{2} c_{bc}^a \epsilon^{\mu\nu} \pi_\mu^b \pi_\nu^c, \quad (44)$$

$$F^\alpha = \frac{1}{2} c_{bc}^{M+\alpha} \epsilon^{\mu\nu} \pi_\mu^b \pi_\nu^c. \quad (45)$$

Now, in general, the covariant divergence of π_μ^a need not vanish. F^α is the H curvature, $F^\alpha = \epsilon^{\mu\nu} (\partial_\mu A_\nu^\alpha + \frac{1}{2} \hat{c}_{\beta\gamma}^\alpha A_\mu^\beta A_\nu^\gamma)$. In terms of the currents π_μ^a , L can be written

$$L = \frac{\kappa}{2} \mathbf{g}_{ab} \pi_\mu^a \pi^{\mu a}. \quad (46)$$

The dual action (in Minkowski space-time) is given in [5]. It is expressed in terms of N scalar fields, χ_a and θ_α , along with the Yang-Mills connection one form A^α , which undergo gauge transformations

$$\delta \chi_a = c_{M+\alpha, a}^b \lambda^\alpha \chi_b \quad (47)$$

$$\delta \theta_\alpha = \hat{c}_{\beta\alpha}^\gamma \lambda^\beta \theta_\gamma \quad (48)$$

$$\delta A^\alpha = d\lambda^\alpha + \hat{c}_{\beta\gamma}^\alpha A^\beta \lambda^\gamma, \quad (49)$$

$\lambda = \lambda^\alpha \hat{e}_\alpha$ being an infinitesimal element of the Lie algebra \mathcal{H} of H . The Lagrangian density is

$$\tilde{L} = -\frac{1}{2}\alpha^{ab}(D_\mu\chi)_a(D^\mu\chi)_b - \frac{1}{2}\epsilon^{\mu\nu}\beta^{ab}(D_\mu\chi)_a(D_\nu\chi)_b - \theta_\alpha F^\alpha, \quad (50)$$

where the covariant derivative of χ_a is defined according to $(D\chi)_a = d\chi_a + c_{a\ M+\alpha}^b A^\alpha \chi_b$, and the dual metric α^{ab} and the antisymmetric tensor β^{ab} are given by

$$\alpha = \left(\kappa \mathbf{g} - \frac{1}{\kappa} \tilde{f} \mathbf{g}^{-1} \tilde{f} \right)^{-1}, \quad \beta = -\frac{1}{\kappa} \mathbf{g}^{-1} \tilde{f} \alpha, \quad (51)$$

where

$$\tilde{f}_{ab} = c_{ab}^c \chi_c + c_{ab}^{M+\alpha} \theta_\alpha, \quad (52)$$

and \mathbf{g} is the group metric projected onto G/H . α in (51) is symmetric by inspection, while antisymmetry for β follows after using the identity $\tilde{f}\alpha\mathbf{g} = \mathbf{g}\alpha\tilde{f}$. Upon varying χ_a , θ_α and A_μ^α and applying identities, one recovers the equations (42), (44) and (45) of the primary system [5]. For this π_μ^a is now defined by

$$\pi_\mu^a = -\alpha^{ab}(D_\mu\chi)_b - \beta^{ab}\epsilon_{\mu\nu}(D^\nu\chi)_b. \quad (53)$$

Substituting this expression back into \tilde{L} and integrating by parts gives

$$\tilde{L} = -\frac{1}{2}\chi_a (D_\mu\pi^\mu)^a + \frac{1}{2}\partial_\mu(\chi_a\pi^{\mu a}) - \theta_\alpha F^\alpha. \quad (54)$$

One can also write

$$\tilde{L} = -\frac{\kappa}{2}\mathbf{g}_{ab}\pi^{\mu a}\pi_\mu^b - \frac{1}{2}\tilde{f}_{ab}\epsilon^{\mu\nu}\pi_\mu^a\pi_\nu^b - \theta_\alpha F^\alpha. \quad (55)$$

Finally, twice (54) minus (55) gives

$$\begin{aligned} \tilde{L} &= \frac{\kappa}{2}\mathbf{g}_{ab}\pi^{\mu a}\pi_\mu^b + \partial_\mu(\chi_a\pi^{\mu a}) \\ &+ \chi_c \left(\frac{1}{2}c_{ab}^c \epsilon^{\mu\nu}\pi_\mu^a\pi_\nu^b - (D_\mu\pi^\mu)^c \right) + \theta_\alpha \left(\frac{1}{2}c_{ab}^{M+\alpha} \epsilon^{\mu\nu}\pi_\mu^a\pi_\nu^b - F^\alpha \right). \end{aligned} \quad (56)$$

The second line vanishes after using the equations of motion [which were the Maurer-Cartan equations (44) and (45) in the primary theory]. Hence, on shell, the dual action agrees with the primary action up to a boundary term.

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